

A congruence involving alternating harmonic sums modulo $p^\alpha q^\beta$

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Abstract In 2014, Wang and Cai established the following harmonic congruence for any odd prime p and positive integer r ,

$$\sum_{\substack{i+j+k=p^r \\ i,j,k \in \mathcal{P}_p}} \frac{1}{ijk} \equiv -2p^{r-1}B_{p-3} \pmod{p^r},$$

where \mathcal{P}_n denote the set of positive integers which are prime to n .

In this note, we obtain the congruences for distinct odd primes p, q and positive integers α, β ,

$$\sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ i \equiv j \equiv k \equiv 1 \pmod{2}}} \frac{1}{ijk} \equiv \frac{7}{8}(2-q)(1-\frac{1}{q^3})p^{\alpha-1}q^{\beta-1}B_{p-3} \pmod{p^\alpha}$$

and

$$\sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{(-1)^i}{ijk} \equiv \frac{1}{2}(q-2)(1-\frac{1}{q^3})p^{\alpha-1}q^{\beta-1}B_{p-3} \pmod{p^\alpha}.$$

Finally, we raise a conjecture that for $n > 1$ and odd prime power $p^\alpha || n$, $\alpha \geq 1$,

$$\sum_{\substack{i+j+k=n \\ i,j,k \in \mathcal{P}_n}} \frac{(-1)^i}{ijk} \equiv \prod_{\substack{q|n \\ q \neq p}} (1-\frac{2}{q})(1-\frac{1}{q^3})\frac{n}{2p}B_{p-3} \pmod{p^\alpha}$$

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and

$$\sum_{\substack{i+j+k=n \\ i,j,k \in \mathcal{P}_n \\ i \equiv j \equiv k \equiv 1 \pmod{2}}} \frac{1}{ijk} \equiv \prod_{\substack{q|n \\ q \neq p}} (1 - \frac{2}{q})(1 - \frac{1}{q^3})(-\frac{7n}{8p})B_{p-3} \pmod{p^\alpha}.$$

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1 Introduction.

Let

$$Z(n) = \sum_{\substack{i+j+k=n \\ i,j,k \in \mathcal{P}_n}} \frac{1}{ijk},$$

where \mathcal{P}_n denote the set of positive integers which are prime to n .

At the beginning of the 21th century, Zhao (Cf.[6]) first announced the following curious congruence involving multiple harmonic sums for any odd prime $p > 3$,

$$Z(p) \equiv -2B_{p-3} \pmod{p}, \quad (1)$$

which holds when $p = 3$ evidently. Here, Bernoulli numbers B_k are defined by the recursive relation:

$$\sum_{i=0}^n \binom{n+1}{i} B_i = 0, n \geq 1.$$

A simple proof of (1) was presented in [2]. Later, Xia and Cai [4] generalized (1) to

$$Z(p) \equiv -\frac{12B_{p-3}}{p-3} - \frac{3B_{2p-4}}{p-4} \pmod{p^2},$$

where $p > 5$ is a prime.

In 2014, Wang and Cai [3] proved for every prime $p \geq 3$ and positive integer r ,

$$Z(p^r) \equiv -2p^{r-1}B_{p-3} \pmod{p^r}. \quad (2)$$

Let $n = 2$ or 4 , for every positive integer $r \geq \frac{n}{2}$ and prime $p > n$, Zhao [5] generalized (2) to

$$\sum_{\substack{i_1+i_2+\dots+i_n=p^r \\ i_1, i_2, \dots, i_n \in \mathcal{P}_p}} \frac{1}{i_1 i_2 \dots i_n} \equiv -\frac{n!}{n+1} p^r B_{p-n-1} \pmod{p^{r+1}}.$$

Recently, for distinct odd primes p, q and positive integers α, β , the authors and Jia[1] proved that

$$Z(p^\alpha q^\beta) \equiv 2(2-q)(1 - \frac{1}{q^3})p^{\alpha-1}q^{\beta-1}B_{p-3} \pmod{p^\alpha}, \quad (3)$$

and conjecture that for $n > 1$, $p^\alpha || n$, $\alpha \geq 1$,

$$Z(n) \equiv \prod_{\substack{q|n \\ q \neq p}} (1 - \frac{2}{q})(1 - \frac{1}{q^3})(-\frac{2n}{p})B_{p-3} \pmod{p^\alpha}.$$

This is the generalization of (2) and (3).

In this paper, we consider the congruences involving alternating harmonic sums and obtain the following theorems.

Theorem 1. *Let p, q be distinct odd primes and α, β positive integer, then*

$$\sum_{\substack{i+j+k=2p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{(-1)^i}{ijk} = -\frac{1}{2}Z(p^\alpha q^\beta) \equiv (q-2)(1 - \frac{1}{q^3})p^{\alpha-1}q^{\beta-1}B_{p-3} \pmod{p^\alpha}.$$

Theorem 2. *Let p, q be distinct odd primes and α, β positive integer, then*

$$\sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{(-1)^i}{ijk} \equiv -\frac{1}{4}Z(p^\alpha q^\beta) \equiv \frac{1}{2}(q-2)(1 - \frac{1}{q^3})p^{\alpha-1}q^{\beta-1}B_{p-3} \pmod{p^\alpha}.$$

Theorem 3. *Let p, q be distinct odd primes and α, β positive integer, then*

$$\sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ i \equiv j \equiv k \equiv 1 \pmod{2}}} \frac{1}{ijk} \equiv \frac{7}{16}Z(p^\alpha q^\beta) \equiv \frac{7}{8}(2-q)(1 - \frac{1}{q^3})p^{\alpha-1}q^{\beta-1}B_{p-3} \pmod{p^\alpha}.$$

Finally, we have the following

Conjecture For any positive integer $n > 1$, if odd prime power $p^\alpha || n$ ($p^\alpha | n$, $p^{\alpha+1} \nmid n$), $\alpha \geq 1$, then

$$\sum_{\substack{i+j+k=n \\ i,j,k \in \mathcal{P}_n}} \frac{(-1)^i}{ijk} \equiv \prod_{\substack{q|n \\ q \neq p}} (1 - \frac{2}{q})(1 - \frac{1}{q^3})\frac{n}{2p}B_{p-3} \pmod{p^\alpha}$$

and

$$\sum_{\substack{i+j+k=n \\ i,j,k \in \mathcal{P}_n \\ i \equiv j \equiv k \equiv 1 \pmod{2}}} \frac{1}{ijk} \equiv \prod_{\substack{q|n \\ q \neq p}} (1 - \frac{2}{q})(1 - \frac{1}{q^3})(-\frac{7n}{8p})B_{p-3} \pmod{p^\alpha}.$$

This is the generalization of Theorem 2 and Theorem 3.

2 Preliminaries.

In order to prove the theorems, we need the following lemmas.

Lemma 1. *Let p, q be distinct odd primes, m positive integer coprime to pq and α, β positive integers, then*

$$\sum_{\substack{i+j+k=mp^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{1}{ijk} \equiv mZ(p^\alpha q^\beta) \pmod{p^\alpha}.$$

Proof. For every triple (i, j, k) of positive integers which satisfies $i + j + k = mp^\alpha q^\beta$, $i, j, k \in \mathcal{P}_{pq}$, we rewrite

$$i = xp^\alpha q^\beta + i_0, \quad j = yp^\alpha q^\beta + j_0, \quad k = zp^\alpha q^\beta + k_0,$$

where $1 \leq i_0, j_0, k_0 < p^\alpha q^\beta$ are prime to pq and integers $x, y, z \geq 0$. Since $3 \leq i_0 + j_0 + k_0 < 3p^\alpha q^\beta$ and $i_0 + j_0 + k_0 = (m - x - y - z)p^\alpha q^\beta$, it is easy to see that

$$\begin{cases} i_0 + j_0 + k_0 = p^\alpha q^\beta \\ x + y + z = m - 1 \end{cases} \quad \text{or} \quad \begin{cases} i_0 + j_0 + k_0 = 2p^\alpha q^\beta \\ x + y + z = m - 2 \end{cases}.$$

Hence

$$\begin{aligned} \sum_{\substack{i+j+k=mp^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{1}{ijk} &= \sum_{\substack{i_0+j_0+k_0=p^\alpha q^\beta \\ x+y+z=m-1, i_0, j_0, k_0 \in \mathcal{P}_{pq}}} \frac{1}{(xp^\alpha q^\beta + i_0)(yp^\alpha q^\beta + j_0)(zp^\alpha q^\beta + k_0)} \\ &+ \sum_{\substack{i_0+j_0+k_0=2p^\alpha q^\beta, 1 \leq i_0, j_0, k_0 < p^\alpha q^\beta \\ x+y+z=m-2, i_0, j_0, k_0 \in \mathcal{P}_{pq}}} \frac{1}{(xp^\alpha q^\beta + i_0)(yp^\alpha q^\beta + j_0)(zp^\alpha q^\beta + k_0)} \\ &\equiv \sum_{\substack{i_0+j_0+k_0=p^\alpha q^\beta \\ x+y+z=m-1, i_0, j_0, k_0 \in \mathcal{P}_{pq}}} \frac{1}{i_0 j_0 k_0} + \sum_{\substack{i_0+j_0+k_0=2p^\alpha q^\beta, 1 \leq i_0, j_0, k_0 < p^\alpha q^\beta \\ x+y+z=m-2, i_0, j_0, k_0 \in \mathcal{P}_{pq}}} \frac{1}{i_0 j_0 k_0} \\ &\equiv \binom{m+1}{2} \sum_{\substack{i_0+j_0+k_0=p^\alpha q^\beta \\ i_0, j_0, k_0 \in \mathcal{P}_{pq}}} \frac{1}{i_0 j_0 k_0} + \binom{m}{2} \sum_{\substack{i_0+j_0+k_0=2p^\alpha q^\beta \\ 1 \leq i_0, j_0, k_0 < p^\alpha q^\beta \\ i_0, j_0, k_0 \in \mathcal{P}_{pq}}} \frac{1}{i_0 j_0 k_0}, \end{aligned} \tag{4}$$

here in the last equation we use the fact that there are $\binom{m+1}{2}$ or $\binom{m}{2}$ triples (x, y, z) of nonnegative integers which satisfy $x + y + z = m - 1$ or $x + y + z = m - 2$, respectively. For the second sum in (4), note that $(i_0, j_0, k_0) \leftrightarrow (p^\alpha q^\beta - i_0, p^\alpha q^\beta - j_0, p^\alpha q^\beta - k_0)$ gives a bijection between the solutions of $i_0 + j_0 + k_0 =$

$p^\alpha q^\beta$ and $i_0 + j_0 + k_0 = 2p^\alpha q^\beta$, where $1 \leq i_0, j_0, k_0 < p^\alpha q^\beta$, thus, we have

$$\begin{aligned} \sum_{\substack{i_0+j_0+k_0=2p^\alpha q^\beta \\ 1 \leq i_0, j_0, k_0 < p^\alpha q^\beta \\ i_0, j_0, k_0 \in \mathcal{P}_{pq}}} \frac{1}{i_0 j_0 k_0} &= \sum_{\substack{i_0+j_0+k_0=p^\alpha q^\beta \\ i_0, j_0, k_0 \in \mathcal{P}_{pq}}} \frac{1}{(p^\alpha q^\beta - i_0)((p^\alpha q^\beta - j_0)(p^\alpha q^\beta - k_0))} \\ &\equiv - \sum_{\substack{i_0+j_0+k_0=p^\alpha q^\beta \\ i_0, j_0, k_0 \in \mathcal{P}_{pq}}} \frac{1}{i_0 j_0 k_0} \pmod{p^\alpha q^\beta}. \end{aligned}$$

Hence, (4) is congruent to

$$\binom{m+1}{2} \sum_{\substack{i_0+j_0+k_0=p^\alpha q^\beta \\ i_0, j_0, k_0 \in \mathcal{P}_{pq}}} \frac{1}{i_0 j_0 k_0} - \binom{m}{2} \sum_{\substack{i_0+j_0+k_0=p^\alpha q^\beta \\ i_0, j_0, k_0 \in \mathcal{P}_{pq}}} \frac{1}{i_0 j_0 k_0} \equiv mZ(p^\alpha q^\beta) \pmod{p^\alpha q^\beta}. \quad (5)$$

Then we complete the proof of Lemma 1. \square

Lemma 2 ([1]). *Let p, q be distinct odd primes and α, β positive integers, if and only if $p = q^2 + q + 1$ or $q = p^2 + p + 1$ or $p|q^2 + q + 1$ and $q|p^2 + p + 1$, we have*

$$Z(p^\alpha q^\beta) \equiv 0 \pmod{p^\alpha q^\beta}.$$

3 Proofs of the theorems.

Proof of Theorem 1. Since $i + j + k = 2p^\alpha q^\beta$ is even, either i, j, k are all even or one of i, j, k is even and the other two are odd, then

$$\sum_{\substack{i+j+k=2p^\alpha q^\beta \\ i, j, k \in \mathcal{P}_{pq}}} \frac{1}{ijk} = \sum_{\substack{i+j+k=2p^\alpha q^\beta, i, j, k \in \mathcal{P}_{pq} \\ i, j, k \text{ are all even}}} \frac{1}{ijk} + \sum_{\substack{i+j+k=2p^\alpha q^\beta, i, j, k \in \mathcal{P}_{pq} \\ \text{exactly one of } i, j, k \text{ is even}}} \frac{1}{ijk}. \quad (6)$$

By symmetry, we have

$$\sum_{\substack{i+j+k=2p^\alpha q^\beta \\ i, j, k \in \mathcal{P}_{pq}}} \frac{(-1)^i}{ijk} = \frac{1}{3} \sum_{\substack{i+j+k=2p^\alpha q^\beta \\ i, j, k \in \mathcal{P}_{pq}}} \frac{(-1)^i + (-1)^j + (-1)^k}{ijk}. \quad (7)$$

If i, j, k are all even, the right hand of (7) equals to

$$\sum_{\substack{i+j+k=2p^\alpha q^\beta, i, j, k \in \mathcal{P}_{pq} \\ i, j, k \text{ are all even}}} \frac{1}{ijk} = \frac{1}{8} \sum_{\substack{i+j+k=p^\alpha q^\beta \\ i, j, k \in \mathcal{P}_{pq}}} \frac{1}{ijk}, \quad (8)$$

where we replace i, j, k by $2i, 2j, 2k$ respectively.

If one of i, j, k is even and the other two are odd, the right hand of (7) equals to

$$-\frac{1}{3} \sum_{\substack{i+j+k=2p^\alpha q^\beta, i,j,k \in \mathcal{P}_{pq} \\ \text{exactly one of } i,j,k \text{ is even}}} \frac{1}{ijk}.$$

Thus, (7) is equal to

$$\begin{aligned} \sum_{\substack{i+j+k=2p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{(-1)^i}{ijk} &= \sum_{\substack{i+j+k=2p^\alpha q^\beta, i,j,k \in \mathcal{P}_{pq} \\ i,j,k \text{ are all even}}} \frac{1}{ijk} - \frac{1}{3} \sum_{\substack{i+j+k=2p^\alpha q^\beta, i,j,k \in \mathcal{P}_{pq} \\ \text{exactly one of } i,j,k \text{ is even}}} \frac{1}{ijk} \\ &= \frac{4}{3} \sum_{\substack{i+j+k=2p^\alpha q^\beta, i,j,k \in \mathcal{P}_{pq} \\ i,j,k \text{ are all even}}} \frac{1}{ijk} - \frac{1}{3} \left(\sum_{\substack{i+j+k=2p^\alpha q^\beta, i,j,k \in \mathcal{P}_{pq} \\ i,j,k \text{ are all even}}} \frac{1}{ijk} \right. \\ &\quad \left. + \sum_{\substack{i+j+k=2p^\alpha q^\beta, i,j,k \in \mathcal{P}_{pq} \\ \text{exactly one of } i,j,k \text{ is even}}} \frac{1}{ijk} \right). \end{aligned}$$

By using (6) and (8), we have

$$\sum_{\substack{i+j+k=2p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{(-1)^i}{ijk} = \frac{1}{6} Z(p^\alpha q^\beta) - \frac{1}{3} \sum_{\substack{i+j+k=2p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{1}{ijk}.$$

It follows from Lemma 1 that

$$\sum_{\substack{i+j+k=2p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{(-1)^i}{ijk} = \frac{1}{6} Z(p^\alpha q^\beta) - \frac{2}{3} Z(p^\alpha q^\beta) = -\frac{1}{2} Z(p^\alpha q^\beta) \pmod{p^\alpha}.$$

By using (3), we complete the proof of Theorem 1. \square

Proof of Theorem 2. For every triple (i, j, k) of positive integers which satisfies $i + j + k = 2p^\alpha q^\beta$, $i, j, k \in \mathcal{P}_{pq}$, we take it into 3 cases.

Cases 1. If $1 \leq i, j, k \leq p^\alpha q^\beta - 1$ are coprime to pq , $(i, j, k) \leftrightarrow (p^\alpha q^\beta - i, p^\alpha q^\beta - j, p^\alpha q^\beta - k)$ is a bijection between the solutions of $i + j + k = 2p^\alpha q^\beta$ and $i + j + k = p^\alpha q^\beta$, $i, j, k \in \mathcal{P}_{pq}$, we have

$$\begin{aligned} \sum_{\substack{i+j+k=2p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ 1 \leq i, j, k \leq p^\alpha q^\beta - 1}} \frac{(-1)^i}{ijk} &\equiv \sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{(-1)^{p^\alpha q^\beta - i}}{(p^\alpha q^\beta - i)(p^\alpha q^\beta - j)(p^\alpha q^\beta - k)} \\ &\equiv \sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{(-1)^i}{ijk} \pmod{p^\alpha}. \end{aligned} \tag{9}$$

Cases 2. If $p^\alpha q^\beta + 1 \leq i \leq 2p^\alpha q^\beta - 1, 1 \leq j, k \leq p^\alpha q^\beta - 1$ are coprime to pq , $(i, j, k) \leftrightarrow (p^\alpha q^\beta + i, j, k)$ is a bijection between the solutions of $i + j + k = 2p^\alpha q^\beta$ and $i + j + k = p^\alpha q^\beta, i, j, k \in \mathcal{P}_{pq}$, we have

$$\begin{aligned} \sum_{\substack{i+j+k=2p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ p^\alpha q^\beta + 1 \leq i \leq 2p^\alpha q^\beta - 1, 1 \leq j, k \leq p^\alpha q^\beta - 1}} \frac{(-1)^i}{ijk} &\equiv \sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{(-1)^{p^\alpha q^\beta + i}}{(p^\alpha q^\beta + i)jk} \\ &\equiv - \sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{(-1)^i}{ijk} \pmod{p^\alpha}. \end{aligned} \quad (10)$$

Cases 3. If $p^\alpha q^\beta + 1 \leq j \leq 2p^\alpha q^\beta - 1, 1 \leq i, k \leq p^\alpha q^\beta - 1$ or $p^\alpha q^\beta + 1 \leq k \leq 2p^\alpha q^\beta - 1, 1 \leq i, j \leq p^\alpha q^\beta - 1$ are coprime to pq , $(i, j, k) \leftrightarrow (i, p^\alpha q^\beta + j, k)$ in the former and $(i, j, k) \leftrightarrow (i, j, p^\alpha q^\beta + k)$ in the later are the bijections between the solutions of $i + j + k = 2p^\alpha q^\beta$ and $i + j + k = p^\alpha q^\beta, i, j, k \in \mathcal{P}_{pq}$, we have

$$\begin{aligned} &\sum_{\substack{i+j+k=2p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ p^\alpha q^\beta + 1 \leq j \leq 2p^\alpha q^\beta - 1, 1 \leq i, k \leq p^\alpha q^\beta - 1}} \frac{(-1)^i}{ijk} + \sum_{\substack{i+j+k=2p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ p^\alpha q^\beta + 1 \leq k \leq 2p^\alpha q^\beta - 1, 1 \leq i, j \leq p^\alpha q^\beta - 1}} \frac{(-1)^i}{ijk} \\ &\equiv \sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{(-1)^i}{i(p^\alpha q^\beta + j)k} + \sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{(-1)^i}{ij(p^\alpha q^\beta + k)} \\ &\equiv 2 \sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{(-1)^i}{ijk} \pmod{p^\alpha}. \end{aligned} \quad (11)$$

Combining (9)-(11), it follows that

$$\begin{aligned} &\sum_{\substack{i+j+k=2p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{(-1)^i}{ijk} = \sum_{\substack{i+j+k=2p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ 1 \leq i, j, k \leq p^\alpha q^\beta - 1}} \frac{(-1)^i}{ijk} + \sum_{\substack{i+j+k=2p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ p^\alpha q^\beta + 1 \leq i \leq 2p^\alpha q^\beta - 1, 1 \leq j, k \leq p^\alpha q^\beta - 1}} \frac{(-1)^i}{ijk} \\ &+ \sum_{\substack{i+j+k=2p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ p^\alpha q^\beta + 1 \leq j \leq 2p^\alpha q^\beta - 1, 1 \leq i, k \leq p^\alpha q^\beta - 1}} \frac{(-1)^i}{ijk} + \sum_{\substack{i+j+k=2p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ p^\alpha q^\beta + 1 \leq k \leq 2p^\alpha q^\beta - 1, 1 \leq i, j \leq p^\alpha q^\beta - 1}} \frac{(-1)^i}{ijk} \\ &\equiv 2 \sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{(-1)^i}{ijk} \pmod{p^\alpha}. \end{aligned}$$

By Theorem 1, we complete the proof of Theorem 2. \square

Proof of Theorem 3. Since $i + j + k = p^\alpha q^\beta$ is odd, either i, j, k are all odd or one of i, j, k is odd and the other two are even, then

$$Z(p^\alpha q^\beta) = \sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ i,j,k \text{ are all odd}}} \frac{1}{ijk} + \sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ \text{exactly one of } i,j,k \text{ is odd}}} \frac{1}{ijk}. \quad (12)$$

By symmetry, similar to Theorem 1, we have

$$\begin{aligned} \sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{(-1)^i}{ijk} &= \frac{1}{3} \sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{(-1)^i + (-1)^j + (-1)^k}{ijk} \\ &= - \sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ i,j,k \text{ are all odd}}} \frac{1}{ijk} + \frac{1}{3} \sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ \text{exactly one of } i,j,k \text{ is odd}}} \frac{1}{ijk} \\ &= -\frac{4}{3} \sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ i,j,k \text{ are all odd}}} \frac{1}{ijk} + \frac{1}{3} \left(\sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ i,j,k \text{ are all odd}}} \frac{1}{ijk} \right. \\ &\quad \left. + \sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ \text{exactly one of } i,j,k \text{ is odd}}} \frac{1}{ijk} \right) \\ &= -\frac{4}{3} \sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ i,j,k \text{ are all odd}}} \frac{1}{ijk} + \frac{1}{3} Z(p^\alpha q^\beta), \end{aligned}$$

where we use (12) in the last equation.

By Theorem 2, we have

$$\sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ i,j,k \text{ are all odd}}} \frac{1}{ijk} \equiv \frac{7}{16} Z(p^\alpha q^\beta) \pmod{p^\alpha}.$$

Therefore, we complete the proof of Theorem 3. \square

Remark 1 By Lemma 2 and Theorem 3 in [1], let p, q be distinct odd primes and α, β positive integers, if and only if $p = q^2 + q + 1$ or $q = p^2 + p + 1$ or $p|q^2 + q + 1$ and $q|p^2 + p + 1$, we have

$$\sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq}}} \frac{(-1)^i}{ijk} \equiv 0 \pmod{p^\alpha q^\beta}$$

and

$$\sum_{\substack{i+j+k=p^\alpha q^\beta \\ i,j,k \in \mathcal{P}_{pq} \\ i \equiv j \equiv k \equiv 1 \pmod{2}}} \frac{1}{ijk} \equiv 0 \pmod{p^\alpha q^\beta}.$$

When $n = pq$, p , q are distinct odd primes, in [1], we have

$$Z(n) \equiv 6\left(1 + \frac{3}{\phi(n) - 2}\right)\left(1 + \frac{1}{(\phi(n) - 1)^3}\right)B_{\phi(n)-2} \pmod{n}.$$

Hence, by Theorem 2 and Theorem 3, we have

$$\sum_{\substack{i+j+k=n \\ i,j,k \in \mathcal{P}_n}} \frac{(-1)^i}{ijk} \equiv -\frac{3}{2}\left(1 + \frac{3}{\phi(n) - 2}\right)\left(1 + \frac{1}{(\phi(n) - 1)^3}\right)B_{\phi(n)-2} \pmod{n}$$

and

$$\sum_{\substack{i+j+k=n \\ i,j,k \in \mathcal{P}_n \\ i \equiv j \equiv k \equiv 1 \pmod{2}}} \frac{1}{ijk} \equiv \frac{21}{8}\left(1 + \frac{3}{\phi(n) - 2}\right)\left(1 + \frac{1}{(\phi(n) - 1)^3}\right)B_{\phi(n)-2} \pmod{n}.$$

In particular, for any odd prime p and positive integer r , we have

$$\sum_{\substack{i+j+k=p^r \\ i,j,k \in \mathcal{P}_p}} \frac{(-1)^i}{ijk} \equiv \frac{1}{2}p^{r-1}B_{p-3} \pmod{p^r}$$

and

$$\sum_{\substack{i+j+k=p^r \\ i,j,k \in \mathcal{P}_p \\ i \equiv j \equiv k \equiv 1 \pmod{2}}} \frac{1}{ijk} \equiv -\frac{7}{8}p^{r-1}B_{p-3} \pmod{p^r}.$$

Remark 2 By the conjecture and Chinese Remainder Theorem, we can count out the remainder of

$$\sum_{\substack{i+j+k=n \\ i,j,k \in \mathcal{P}_n}} \frac{(-1)^i}{ijk} \quad \text{and} \quad \sum_{\substack{i+j+k=n \\ i,j,k \in \mathcal{P}_n \\ i \equiv j \equiv k \equiv 1 \pmod{2}}} \frac{1}{ijk}$$

modulo n for any positive integer n .

Problem 1 Can we find arithmetical functions $f(n)$ and $g(n)$ such that

$$\sum_{\substack{i+j+k=n \\ i,j,k \in \mathcal{P}_n}} \frac{(-1)^i}{ijk} \equiv f(n) \pmod{n} \quad \text{and} \quad \sum_{\substack{i+j+k=n \\ i,j,k \in \mathcal{P}_n \\ i \equiv j \equiv k \equiv 1 \pmod{2}}} \frac{1}{ijk} \equiv g(n) \pmod{n}.$$

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